A Time Parallel Algorithm Based ParaExp for Optimality Systems

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Abstract

We propose a new parallel-in-time algorithm for solving optimal control problems constrained by partial differential equations. Our approach, which is based on a deeper understanding of ParaExp, considers an overlapping time-domain decomposition in which we combine the solution of homogeneous problems using exponential propagation with the local solutions of inhomogeneous problems. The algorithm yields a linear system whose matrix-vector product can be fully performed in parallel. We then propose a preconditioner to speed up the convergence of GMRES in the special cases of the heat and wave equations.

Optimal control problem

Let us consider the following linear-quadratic optimal control problem

$$\min_{\nu} \frac{1}{2}\|y(T)-y_{tg}\|^2 + \frac{\alpha}{2}\int_0^T \|\nu(t)\|^2\,dt,$$
$$\text{s.t.} \qquad \dot{y}(t) = \mathcal{L}y(t) + \nu(t),\ y(0) = y_{in},\ t \in (0,T), \tag{1}$$

where α is a regularization parameter, y_{in} the initial condition and y_{tg} the target state. Introducing the adjoint state λ we obtain the following the optimality system

$$\dot{y} = \mathcal{L}y - \frac{1}{\alpha}\lambda, \qquad \dot{\lambda} = -\mathcal{L}^T\lambda, \tag{2}$$

with initial and final condition $y_{in} = y(0)$ and $\lambda(T) = y(T) - y_{tg}$ respectively. We solve this system using a parallel in time idea based ParaExp algorithm presented bellow.

Time Parallel Algorithm

For $L \in \mathbb{N}^*$, we consider the non-overlapping sub-intervals $(T_{\ell-1}, T_\ell)$, $\ell = 1,\dots,L$ of $(0,T)$ with $T_\ell = \ell\Delta T$ and $\Delta T = T/L$. We define $Y_\ell \approx y(T_\ell)$ and $\Lambda_\ell \approx \lambda(T_\ell)$. We now consider the decomposition of the couple of the ODEs (2) into homogeneous and inhomogeneous parts over sub-intervals.

Homegeneous sub-problems on λ : Backward exponential propagation

$$\dot{\lambda}_\ell(t) = -\mathcal{L}^T\lambda_\ell(t),\ \lambda_\ell(T_L) = \Lambda_L, \text{on } (T_{\ell-1}, T_L), \ell = 1,\dots,L. \tag{3}$$

Inhomegeneous sub-problems on y : Forward integration

$$\dot{w}_\ell(t) = \mathcal{L}w_\ell(t) - \frac{1}{\alpha}\lambda_\ell(t),\ \ w_\ell(T_{\ell-1}) = 0,\ \text{on } (T_{\ell-1}, T_\ell), \ell = 1,\dots,L \tag{4}$$

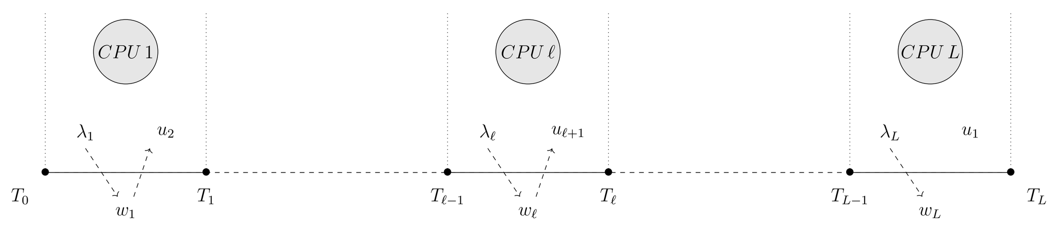
Homegeneous sub-problems on y : Forward exponential propagation:

$$\begin{aligned}\dot{u}_1(t) &= \mathcal{L}u_1(t),\ u_1(T_0) = y_{in},\ \text{on } (T_0, T_L)\\ \dot{u}_\ell(t) &= \mathcal{L}u_\ell(t),\ u_\ell(T_\ell) = w_{\ell-1}(T_{\ell-1}),\ \text{on } (T_{\ell-1}, T_L), \ell = 2,\dots,L\end{aligned} \tag{5}$$

Optimal trajectory : By superposition principle ,

$$Y_\ell = w_\ell(T_\ell) + \sum_{j=1}^{\ell} u_j(T_\ell) \quad \text{on } (T_{\ell-1}, T_\ell), \quad \ell = 1,\dots,L. \tag{6}$$

Parallel technic



Let $\mathcal{Q}_\ell$ and $\mathcal{P}_\ell$ be exponential propagators that solve the homogeneous sub-problems (3) and (5) on $(T_{\ell-1}, T_L)$ respectively and $\mathcal{R}_\ell$ the solver of the inhomoegeneous sub-problems (4) on $(T_{\ell-1}, T_\ell)$. Then

$$Y_\ell = \mathcal{P}_1(T_\ell)\cdot y_{in} - \frac{1}{\alpha}\mathcal{R}_\ell(T_\ell)\cdot\Lambda_L - \frac{1}{\alpha}\sum_{j=2}^{\ell}\mathcal{P}_j(T_\ell)\cdot\mathcal{R}_{j-1}(T_{j-1})\cdot\Lambda_L,\ \ell = 1,\dots,L,$$

and $\Lambda_\ell = \mathcal{Q}_\ell(T_\ell)\cdot\Lambda_L$, $\ell = 1,\dots,L-1$. Using the final condition $\Lambda_L = Y_L - y_{tg}$, we find that Λ_L satisfies

$$M\cdot\Lambda_L + \mathbf{b} = 0, \tag{7}$$

where $\mathbf{b} := y_{tg} - \mathcal{P}_1(T_L)\cdot y_{in}$ and

$$M := I + \frac{1}{\alpha}\mathcal{R}_L(T_L) + \frac{1}{\alpha}\sum_{j=2}^{L}\mathcal{P}_j(T_L)\cdot\mathcal{R}_{j-1}(T_{j-1}), \qquad I: \ \text{identity matrix.}$$

One can use GMRES to solve (7). Since the matrix M is generally ill-conditioned, an appropriate preconditioner is needed, which we construct for the heat and wave equations.

Preconditioner : Heat equation case

We consider the optimal control problem involving the following heat equation

$$\dot{y} = \Delta y + \nu \text{ on } (0,1)\times(0,T),\ y(x,0) = y_{in}(x),\ y(0,t) = y(1,t) = 0,\ \ T \geq 1. \tag{8}$$

Finite difference discretization in space : $\dot{y}(t) = \mathcal{L}y(t) + \nu(t),\ y(0) = y_{in}$ where $y(t), y_0, \nu(t) \in \mathbb{R}^r$, r the number of points in space.

Preconditioner : $P^{-1} = \mathcal{L}(\mathcal{L} - \frac{1}{2\alpha}I)^{-1}$ for solving (7) derived from a spectral analysis of the continuous form of M since $\mathcal{L}$ is symmetric.

Application of P^{-1} : each application of P^{-1} involves only one solution of the elliptic problem $(\mathcal{L} - \frac{1}{2\alpha}I)v = \mathbf{f}$.

Therefore, if an algebraic method is used for computing $\mathcal{Q}_\ell$ and $\mathcal{P}_\ell$ then we have the following result.

Theorem 1 *Let N be given and $\mathcal{R}_\ell$ be approximated using implicit Euler with N fine sub-intervals over each $(T_{\ell-1}, T_\ell)$. Then any eigenvalue μ of MP^{-1} satisfies*

$$1 < \mu < 1 + \frac{\delta t}{\alpha}, \qquad \delta t = T/LN.$$

The preconditioner P^{-1} is more efficient for a high-order approximation of $\mathcal{R}_\ell$ than for a lower order one. We solve (7) in GMRES for $N = 1000, r = 100$ and $\alpha = 10^{-6}$. We use for this instance `gmres` in Matlab with `restart=[], tol=1e-8` and `maxit=size(M,1)`. SDIRK is a quadrature formula of stage $c = (1/2 + \sqrt(3)/6, 1/2 - \sqrt(3)/6)$ and weights $d = (1/2, 1/2)$.

Schemes	Cond(M)	#Iters (M)	Cond(MP^{-1})	#Iters (MP^{-1})
Euler	4.97e2	500	7.76	44
SDIRK	5.13e3	500	1.35	4

Preconditioner : Wave equation case

We now consider the optimal control problem involving the following wave equation

$$u_{tt} = \Delta u + \nu \text{ on } (0,1)\times(0,T),\ u(x,0) = u_0(x),\ \partial_t u(x,0) = 0. \tag{9}$$

ODE system from (9) : $\dot{y} = \mathcal{L}y + \mathcal{B}\nu$ where $y = \begin{bmatrix} u \\ u_t\end{bmatrix}$, $\mathcal{L} = \begin{bmatrix} 0 & I \\ \Delta & 0\end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 \\ I\end{bmatrix}$.

Discretization of Δ : We use finite difference with Dirichlet boundary condition.

Continuous form of M : Since $\mathcal{L}$ is no longer symmetric the continuous analogue of M takes the form

$$\mathcal{M} := I + \frac{1}{\alpha}\int_0^T \exp(s\mathcal{L})\mathcal{B}\mathcal{B}^T\exp(s\mathcal{L}^T)\,ds = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix},$$

where for $A = -\Delta, M_{11} = I + \frac{T}{2\alpha}A^{-1} - \frac{1}{4\alpha}A^{-3/2}\sin(2TA^{1/2}), M_{12} = \frac{1}{2\alpha}A^{-1}(I - \cos(2TA^{1/2}))\ , M_{21} = M_{12}$ and $M_{22} = (1 + \frac{T}{2\alpha})I + \frac{1}{4\alpha}A^{-1/2}\sin(2TA^{1/2})$.

Preconditioner : $P^{-1} = I - \begin{bmatrix} (aI + bA)^{-1} & 0 \\ 0 & cI\end{bmatrix}$, $a = T, b = 2\alpha$ and finally $c = 1/(2\alpha + T)$. This is derived from the spectral analysis of $\mathcal{M}$ based A.

Application : Each application of P^{-1} therefore involves only one solution of the elliptic problem $(aI + bA)v = \mathbf{f}$.

We solve (7) in GMRES for $N = 1000, r = 100$ and $\alpha = 10^{-6}$. We use for this instance the function `gmres` in Matlab with `restart=[], tol=1e-8` and `maxit=size(M,1)`.

Schemes	Cond(M)	#Iters (M)	Cond(MP^{-1})	#Iters (MP^{-1})
Euler	3.8e4	84	2.59	4
SDIRK	3.8e4	84	2.59	4

Conclusion

We introduced a new time parallel algorithm for time dependent linear quadratic optimal control problem when a cheap exponential integrator is available. We proposed two preconditioners to solves efficiently the linear equation that comes from the algorithm on the particular cases of heat and wave equations respectively. We are now studying the optimal control problem involving the wave typ equation with boundary control in one and several dimensions including CFL conditions cases.

References

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